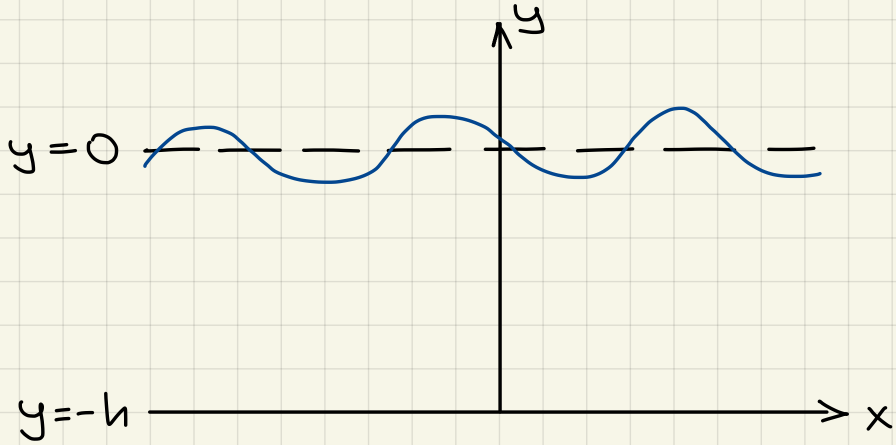
III. Waves

3. 1.



$$\bar{u} = u(x,y,t)\,\bar{e}_x + v(x,y,t)\,\bar{e}_y$$

$$\bar{\omega} = (\partial_x v - \partial_y u)\,\bar{e}_z = 0 \quad , \quad \nu = 0$$

irrot. flow ideal fluid

$$\bar{u} = \bar{\nabla}\varphi(x,y,t) \quad \Rightarrow \quad u = \frac{\partial\varphi}{\partial x} \;, \; v = \frac{\partial\varphi}{\partial y}$$

$$\bar{\nabla}\cdot\bar{u} = \underbrace{\bar{\nabla}^2\varphi = 0}_{\text{incompr. flow}}$$

$$y = \xi(x,t)$$

1. Kinem. cond.

$$y - \xi(x,t) = 0$$

$$\frac{D}{Dt}\left[y - \xi(x,t)\right] = v - \frac{\partial\xi}{\partial t} - (\bar{u}\cdot\bar{\nabla})\,\xi$$

$$= v - \frac{\partial\xi}{\partial t} - u\frac{\partial\xi}{\partial x} = 0$$

$$\frac{\partial\xi}{\partial t} + u\frac{\partial\xi}{\partial x} = v$$

2. Bernoulli law

$$\frac{\partial\bar{\nabla}\varphi}{\partial t} + \bar{\nabla}\left(\frac{p}{\rho} + \frac{1}{2}\bar{u}^2 + \chi\right) = 0$$

$$\frac{\partial\varphi}{\partial t} + \frac{p}{\rho} + \frac{1}{2}\bar{u}^2 + \chi = G(t)$$

$$\bar{u} = \bar{\nabla}(\varphi + s(t)) = \bar{\nabla}\varphi$$

$G = p_0/\rho$, p_0 – atm. pres.

B. law at the surf.:

$$\frac{\partial \varphi}{\partial t} + \frac{1}{2}\bar{u}^2 + \chi = \underbrace{\frac{\partial \varphi}{\partial t} + \frac{1}{2}\bar{u}^2 + g\xi = 0}$$

Assumption: displacement is small w. respect to wavelength.

Δ - small param., $\Delta \ll L$.

$$\xi \sim \Delta \quad , \quad u \sim \Delta \quad , \quad v \sim \Delta$$

$$\frac{\partial \xi}{\partial t} + u\frac{\partial \xi}{\partial x} = v$$

$$\left[\frac{\partial \xi}{\partial t} + u\frac{\partial \xi}{\partial x}\right](x, y, t) = \Big[\overset{\sim\Delta}{\frac{\partial \xi}{\partial t}} + \overbrace{u\frac{\partial \xi}{\partial x}}^{\sim\Delta^2}\Big](x,0,t) + O(\Delta^2)$$

$$\approx \frac{\partial \xi}{\partial t}(x, 0, t)$$

$$v(x, y, t) \approx v(x, 0, t)$$

linearized k. cond.: $\underbrace{\frac{\partial \xi}{\partial t} = v \quad \text{at} \quad y = 0}$

$$\frac{\partial \varphi}{\partial t} + \frac{1}{2}\bar{u}^2 + g\xi \to \underbrace{\frac{\partial \varphi}{\partial t} + g\xi = 0 \quad \text{at } y = 0}$$

Assumpt.: $\xi(x,t) = A\cos(kx - \omega t)$

$$\Rightarrow \frac{\partial \xi}{\partial t} = v = \frac{\partial \varphi}{\partial y} = \omega A \sin(kx - \omega t)$$

$$\frac{\partial \varphi}{\partial t} = -g\xi = -gA\cos(kx - \omega t)$$

$$\Rightarrow \varphi(x, y, t) = f(y)\sin(kx - \omega t)$$

$$\nabla^2 \varphi = -k^2 f \sin(\ldots) + f'' \sin(\ldots) = 0$$

$$\Rightarrow f'' - k^2 f = 0$$

$$f(y) = Be^{ky} + Ce^{-ky}$$

$y = -h$ no penetration of the wall

$$v = \frac{\partial \Phi}{\partial y} = 0 \Rightarrow f'(-h) = kBe^{-kh} - kCe^{kh} = 0$$

$$\Rightarrow B = Ce^{2kh} \qquad (1')$$

$y = 0$

$$\frac{\partial \zeta}{\partial t} = \frac{\partial \Phi}{\partial y} \Rightarrow (kB - kC)\sin(kx - \omega t) = \omega A \sin(kx - \omega t)$$

$$\Rightarrow B - C = \frac{\omega A}{k} \qquad (2')$$

$$\frac{\partial \Phi}{\partial t} = -g\zeta \Rightarrow -(B + C)\,\omega\cos(kx - \omega t) = -gA\cos(kx - \omega t)$$

$$\Rightarrow B + C = \frac{gA}{\omega} \qquad (3')$$

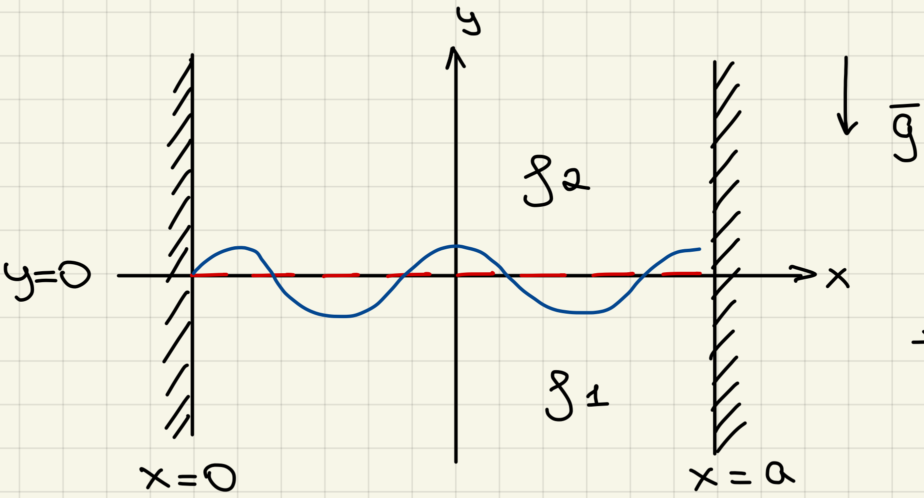
$$\frac{B - C}{B + C} = \frac{C(e^{2kh} - 1)}{C(e^{2kh} + 1)} = \frac{\omega A}{k}\,\frac{\omega}{gA}$$

$$\underbrace{\frac{\cancel{e^{kh}}(e^{kh} - e^{-kh})}{\cancel{e^{kh}}(e^{kh} + e^{-kh})}}_{\tanh(kh)} = \frac{\omega^2}{kg}$$

$$\Downarrow$$

$$\omega^2 = kg\tanh(kh), \qquad k = 2\pi/L$$

3.3



+ surf tension T

eq-n of the interf.

$$\xi(x,t) = A_N(x)\cos(\omega_N t + \varepsilon_N),$$
$$N = 1, 2, \dots$$

1. Bern.

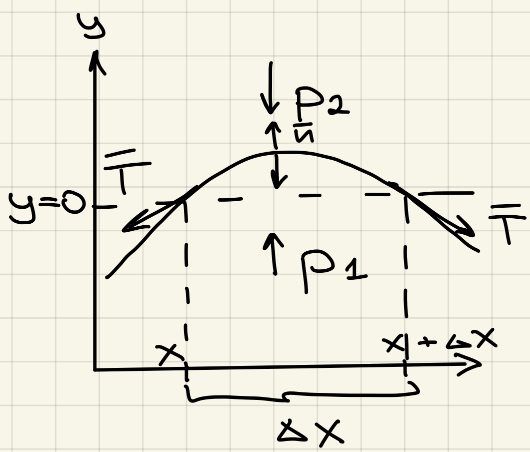
$$\frac{\partial \varphi_i}{\partial t} + \frac{p_i}{\rho_i} + \frac{1}{2}\bar{u}_i^2 + \chi = \underbrace{G_i(t)}_{\text{arbitrary}}$$

$$\rho_1 G_1 = \rho_2 G_2$$

At the interf.:

$$\rho_1\left(\frac{\partial \varphi_1}{\partial t} + \frac{1}{2}\bar{u}_1^2 + g\xi\right) + p_1$$
$$= \rho_2\left(\frac{\partial \varphi_2}{\partial t} + \frac{1}{2}\bar{u}_2^2 + g\xi\right) + p_2$$

2.



net pressure force

$$-p_2\Delta x + p_1\Delta x$$

Surf. tension: $T\left.\frac{\partial \xi}{\partial x}\right|_{x+\Delta x} - T\left.\frac{\partial \xi}{\partial x}\right|_{x}$

$$= T\Delta x\,\frac{\left.\frac{\partial \xi}{\partial x}\right|_{x+\Delta x} - \left.\frac{\partial \xi}{\partial x}\right|_{x}}{\Delta x} = T\Delta x\,\frac{\partial^2 \xi}{\partial x^2}$$

$$p_1 - p_2 + T\frac{\partial^2 \xi}{\partial x^2} = 0$$

$$T \frac{\partial^2 \zeta}{\partial x^2} = p_2 - p_1$$

$$\rho_1 \left(\frac{\partial \varphi_1}{\partial t} + \frac{1}{2} \bar{u}_1^2 + g\zeta \right)$$
$$= \rho_2 \left(\frac{\partial \varphi_2}{\partial t} + \frac{1}{2} \bar{u}_2^2 + g\zeta \right) + T \frac{\partial^2 \zeta}{\partial x^2}$$

Linearization ($|\bar{u}_i| \sim \varphi_i \sim \zeta \sim \Delta \ll L$):

$$\rho_1 \left(\frac{\partial \varphi_1}{\partial t} + g\zeta \right) = \rho_2 \left(\frac{\partial \varphi_2}{\partial t} + g\zeta \right) + T \frac{\partial^2 \zeta}{\partial x^2}$$

3. $$\frac{\partial \zeta}{\partial t} = v_i = \frac{\partial \varphi_i}{\partial y} \quad \text{at } y = 0$$

$$\zeta(x,t) = A_N(x) \cos(\omega_N t + \varepsilon_N)$$

$$\frac{\partial \varphi_i}{\partial y} = \frac{\partial \zeta}{\partial t} = - A_N(x) \omega_N \sin(\omega_N t + \varepsilon_N)$$

$$\Rightarrow \varphi_i(x,y,t) = f_1^i(x) f_2^i(y) \sin(\omega_N t + \varepsilon_N)$$

Let's subs. into $\nabla^2 \varphi = 0$ (mass conserv.):

$$f_1'' f_2 + f_1 f_2'' = 0$$

$$\frac{f_1''}{f_1} = - \frac{f_2''}{f_2} = K$$

Solve for $f_1(x)$: $f_1'' - K f_1 = 0$

1) $K > 0 \rightarrow f_1(x) = A e^{\sqrt{K} x} + B e^{-\sqrt{K} x}$

No penetration at $x = 0$ and $x = a$

$$\Rightarrow u = \frac{\partial \varphi_i}{\partial x} = 0$$

$$\Rightarrow \quad A = B = 0.$$

2) $K = 0 \Rightarrow A = B = 0.$

3) $K < 0$, $K = -\lambda^2 < 0.$

$$f_1(x) = A \cos \lambda x + B \sin \lambda x$$

$$f_1'(0) = \lambda B = 0$$

$$f_1'(a) = -\lambda A \underline{\sin \lambda a = 0}$$

$$\Rightarrow \lambda = \frac{N\pi}{a}, \quad N = 1, 2, \ldots$$

$$f_1(x) = A \cos\left(\frac{N\pi x}{a}\right)$$

For $f_2(y) = C e^{\frac{N\pi}{a} y} + D e^{-\frac{N\pi}{a} y}$

At $y \to -\infty$: $\varphi_1 = 0$

At $y \to +\infty$: $\varphi_2 = 0$

For fluid 1:

$$\varphi_1(x, y, t) = \tilde{A} \cos\left(\frac{N\pi x}{a}\right) e^{\frac{N\pi}{a} y} \sin(\omega_N t + \varepsilon_N),$$

$$\tilde{A} = AC.$$

For fluid 2:

$$\varphi_2(x, y, t) = \tilde{B} \cos\left(\frac{N\pi x}{a}\right) e^{-\frac{N\pi}{a} y} \sin(\omega_N t + \varepsilon_N),$$

$$\tilde{B} = AD.$$

Kin. cond. at $y = 0$: $\frac{\partial \zeta}{\partial t} = \frac{\partial \varphi_1}{\partial y} = \frac{\partial \varphi_2}{\partial y}$

$$\cancel{\frac{N\pi}{a}} \tilde{A} \cancel{\cos\left(\frac{N\pi x}{a}\right)} \sin(\cancel{\omega_N t + \varepsilon_N}) = -\cancel{\frac{N\pi}{a}} \tilde{B} \cancel{\cos(\ldots)} \times \cancel{\sin(\ldots)}$$

$$\Rightarrow \tilde{A} = -\tilde{B}$$

$$-\omega_N A_N(x) \cancel{\sin(\omega_N t + \varepsilon_N)} = \frac{N\pi}{a} \tilde{A} \cos\left(\frac{N\pi x}{a}\right) \cancel{\sin(\ldots)}$$

$$A_N(x) = -\frac{N\pi}{\omega_N a} \tilde{A} \cos\left(\frac{N\pi x}{a}\right) = \tilde{A}_N \cos\left(\frac{N\pi x}{a}\right)$$

$$\tilde{A}_N = -\frac{N\pi}{\omega_N a} \tilde{A} = \text{const}$$

$$\zeta(x,t) = \tilde{A}_N \cos\left(\frac{N\pi x}{a}\right)\cos(\omega_N t + \varepsilon_N)$$

at $y=0$

$$\rho_1\left(\frac{\partial \varphi_1}{\partial t} + g\zeta\right) = \rho_2\left(\frac{\partial \varphi_2}{\partial t} + g\zeta\right) + T\frac{\partial^2 \zeta}{\partial x^2}$$

$$\rho_1\Big[\cancel{\tilde{A}}\,\omega_N \cancel{\cos\left(\frac{N\pi x}{a}\right)}\cancel{\cos(\omega_N t + \varepsilon_N)} - g\frac{N\pi}{\omega_N a}\cancel{\tilde{A}}\cancel{\cos\left(\frac{N\pi x}{a}\right)}$$
$$\times \cancel{\cos(\omega_N t + \varepsilon_N)}\Big] = \rho_2\Big[-\cancel{\tilde{A}}\,\omega_N \cancel{\cos\left(\frac{N\pi x}{a}\right)}$$
$$\times \cancel{\cos(\omega_N t + \varepsilon_N)} - g\frac{N\pi}{\omega_N a}\cancel{\tilde{A}}\cancel{\cos\left(\frac{N\pi x}{a}\right)}\cancel{\cos(\omega_N t + \varepsilon_N)}\Big]$$
$$+ T\frac{N\pi}{\omega_N a}\frac{N^2\pi^2}{a^2}\cancel{\tilde{A}}\cancel{\cos\left(\frac{N\pi x}{a}\right)}\cancel{\cos(\omega_N t + \varepsilon_N)}$$

$$\rho_1\left(\omega_N - g\frac{N\pi}{\omega_N a}\right) = \rho_2\left(-\omega_N - g\frac{N\pi}{\omega_N a}\right) + T\frac{N\pi}{\omega_N a}\frac{N^2\pi^2}{a^2}$$

$$\omega_N^2(\rho_1 + \rho_2) = g\frac{N\pi}{a}(\rho_1 - \rho_2) + T\frac{N\pi}{a}\frac{N^2\pi^2}{a^2}$$

$$\omega_N^2(\rho_1 + \rho_2) = \frac{N\pi}{a}\left[g(\rho_1 - \rho_2) + T\frac{N^2\pi^2}{a^2}\right]$$

disp. relation

$\rho_1 > \rho_2 \rightarrow \omega_N$ is real. $\rightarrow$ stability $\sim \cos(\omega_N t)$

$\rho_1 < \rho_2 \longrightarrow \omega_N$ can be imaginary

$\omega_N = i\gamma_N \sim \cos(i\gamma_N t)$

$$\cos(x + iy) = \cos x \cosh y + \sin x \sinh y$$

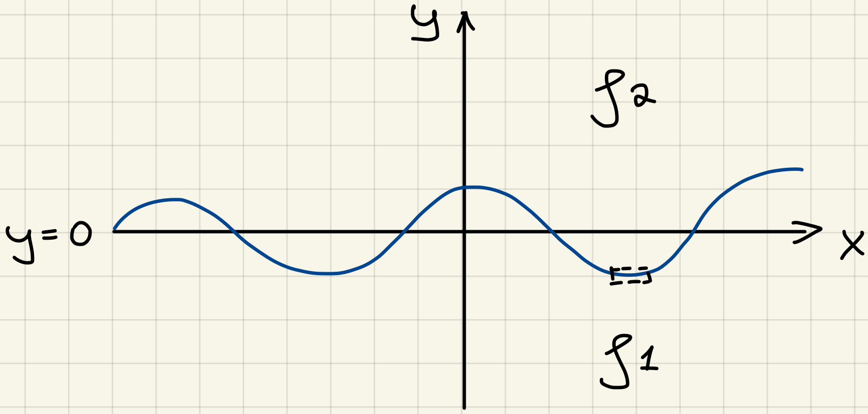
$$\cos(i\gamma_N t) = \cosh(\gamma_N t)$$

unbounded growth of dist. in time

$\rightarrow$ instability.

$$T\frac{N^2\pi^2}{a^2} > g|(\rho_1 - \rho_2)| \quad \text{— stability condition}$$

3.2



$\rho_1 > \rho_2$

No surf. tension.

We assume $\xi(x,t) = A\cos(kx - \omega t)$

1. Balance of forces

$$p_1 = p_2$$

2. Bernoulli

$$\frac{\partial \varphi_i}{\partial t} + \frac{p_i}{\rho_i} + \frac{1}{2}\bar{u}_i^2 + \chi = G_i(t), \text{ where } i = 1,2$$

$$\rho_1 G_1 = \rho_2 G_2$$

$$\rho_1\left(\frac{\partial \varphi_1}{\partial t} + \bar{u}_1^2 + \chi\right) + p_1 = \rho_2\left(\frac{\partial \varphi_2}{\partial t} + \bar{u}_2^2 + \chi\right) + p_2$$

At the surf.:

$$\rho_1\left(\frac{\partial \varphi_1}{\partial t} + g\xi\right) = \rho_2\left(\frac{\partial \varphi_2}{\partial t} + g\xi\right) \text{ at } y=0$$

3. Kinem. cond.

$$\frac{\partial \xi}{\partial t} = v_i = \frac{\partial \varphi_i}{\partial y}$$

Form of φ_i:

$$\frac{\partial \varphi_i}{\partial y} = A\omega \sin(kx - \omega t)$$

$$\Rightarrow \varphi_i = f_i(y)\sin(kx - \omega t)$$

$$\nabla^2 \varphi_i = 0 \Rightarrow -k^2 f_i + f_i'' = 0$$

$$\varphi_1(x,y,t) = (Be^{ky} + Ce^{-ky}) \sin(kx - \omega t)$$

$$\varphi_2(x,y,t) = (De^{ky} + Ee^{-ky}) \sin(kx - \omega t)$$

At $y \to -\infty$ $\varphi_1 = 0 \Rightarrow C = 0$

At $y \to \infty$ $\varphi_2 = 0 \Rightarrow D = 0$

$$\varphi_1 = Be^{ky} \sin(\dots), \varphi_2 = Ee^{-ky} \sin(\dots)$$

Now let's exploit kinem. cond at $y = 0$:

$$\frac{\partial \zeta}{\partial t} = A\omega \sin(kx - \omega t) = \frac{\partial \varphi_1}{\partial y} = \frac{\partial \varphi_2}{\partial y}$$

$$\Rightarrow kBe^{ky} = -kEe^{-ky} = A\omega$$

$$\Rightarrow B = -Ee^{-2ky}$$

and $-kEe^{-ky} = A\omega \Rightarrow E = -\frac{\omega}{k} A e^{ky}$

$$\Rightarrow \varphi_1 = -Ee^{-ky} \sin(\dots) = \frac{\omega}{k} A \sin(\dots)$$

$$\varphi_2 = Ee^{-ky} \sin(\dots) = -\frac{\omega}{k} A \sin(\dots)$$

$$\rho_1 \left(-\frac{\omega^2}{k} A \cancel{\cos(\dots)} + gA \cancel{\cos(\dots)}\right)$$

$$= \rho_2 \left(\frac{\omega^2}{k} A \cancel{\cos(\dots)} + gA \cancel{\cos(\dots)}\right)$$

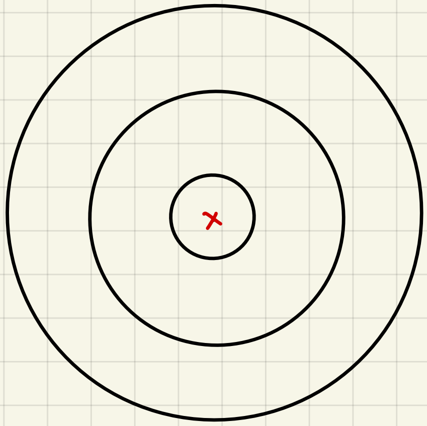
$$\frac{\omega^2}{k}(\rho_2 + \rho_1) = g(\rho_1 - \rho_2)$$

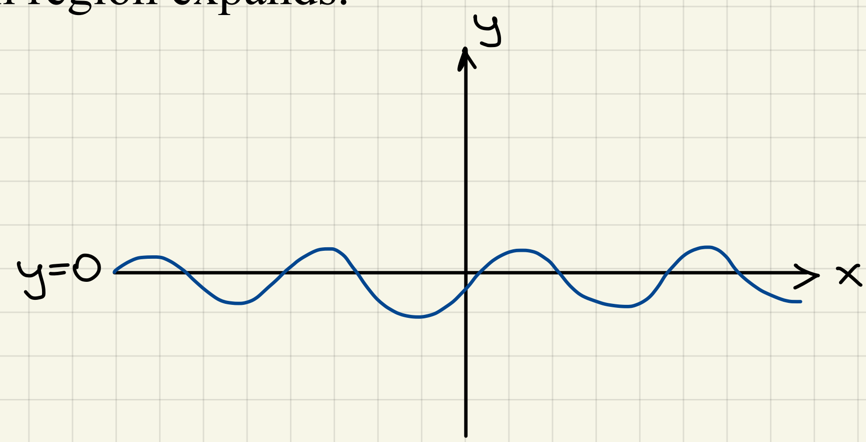
$$\Rightarrow \boxed{\omega^2 = gk \frac{\rho_1 - \rho_2}{\rho_1 + \rho_2}}$$ dispersion relation

If $\rho_1 > \rho_2 \to \omega$ is real.

Note that in the book $\omega^2 \sim |k| \longrightarrow$ if you assume $\zeta \sim \sin(-kx - \omega t)$ (wave propagate in the opposite direction) $\longrightarrow$ the same disp. relation.

3.5 When a stone is dropped into a deep pond, waves are eventually observed only beyond a central region of calm water which expands in radius with time. Furthermore, the wavelength just beyond this calm region is constant, about 4.5 cm.
Use 2D plane wave theory, including both gravity and surface tension, to account broadly for these observations, and obtain an estimate for the speed at which the calm region expands.





1) Balance forces

$$p_A - p = T \frac{\partial^2 \zeta}{\partial x^2} \quad \text{at} \quad y = \zeta(x,t)$$

2) $$\frac{\partial \varphi}{\partial t} + \frac{1}{2}\bar{u}^2 + \frac{p}{\rho} + \chi = G(t)$$

Assume $G = p_A/\rho$

At the linear. surf.: $$\frac{\partial \varphi}{\partial t} + g\zeta = \frac{p_A}{\rho} - \frac{p}{\rho} = \frac{T}{\rho}\frac{\partial^2 \zeta}{\partial x^2}$$
at $y=0$.

$$\underbrace{L(\varphi) = 0}_{\text{Tayl. ser. } y=0} \quad \text{at} \quad y = \zeta(x,t)$$

Assumption : $\zeta(x,t) = A\cos(kx - \omega t)$

$\rightarrow$ use lin. kinem. cond. $\frac{\partial \zeta}{\partial t} = \frac{\partial \varphi}{\partial y}$

$\rightarrow \varphi(x,y,t) = f(y)\sin(kx - \omega t)$

$$\varphi = (B e^{ky} + \cancel{C e^{-ky}}) \sin(kx - \omega t)$$

At $y \to -\infty \to \varphi = 0 \Rightarrow C = 0.$

$$\frac{\partial \zeta}{\partial t} = A\omega \sin(\dots) = \frac{\partial \varphi}{\partial y} = kB\sin(\dots)$$

at $y = 0$

$$\Rightarrow B = \frac{\omega}{k} A$$

Bernoulli: $-\frac{\omega^2}{k}\cancel{A}\,\cancel{\cos(kx-\omega t)} + g\cancel{A}\,\cancel{\cos(kx-\omega t)}$

$$= -\frac{T}{\rho} k^2 \cancel{A}\,\cancel{\cos(kx-\omega t)}$$

$$\underbrace{\omega^2 = \underbrace{gk}_{\text{for deep water}} + \frac{Tk^3}{\rho}}_{\text{disp. rel.}}$$

Speed at which wavepacket propag. away from where you've dropped a stone.

$$c_g = \frac{d\omega}{dk}$$

$$\frac{d\omega^2}{dk} = 2\omega c_g = g + \frac{2Tk^2}{\rho}$$

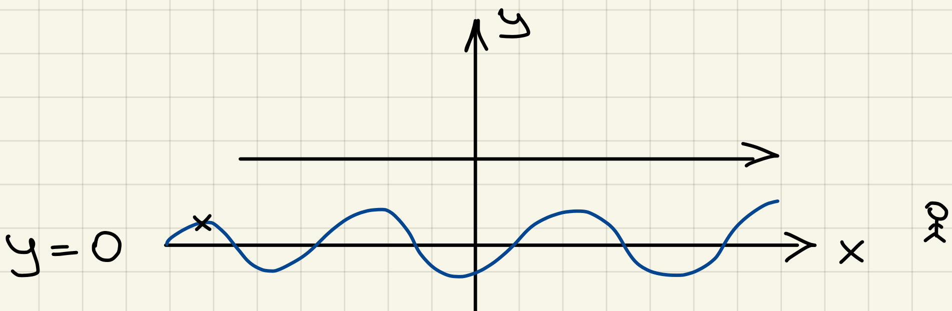
$$\Rightarrow c_g = \frac{g + \frac{2Tk^2}{\rho}}{2\sqrt{gk + \frac{Tk^3}{\rho}}} = c_g(k)$$

$g = 9.81\,\frac{m}{s^2}$, $\rho = 10^3\,\frac{kg}{m^3}$, $T = 0.074\,\frac{N}{m}$

Let's evaluate c_g for max. wavelength $\sim \frac{1}{k}$

$$\frac{dc_g}{dk} = 0$$

3.8 Surface waves generated by a mid-Atlantic storm arrive at the British coast with period 15 seconds. A day later the period of the waves arriving has dropped to 12.5 seconds. Roughly how far away did the storm occur?



$$\xi(x,t) = A\cos(kx-\omega t)$$

Our model — surface waves on deep water.

At $y=0$:

1) Lin. kinem. cond.: $\frac{\partial \xi}{\partial t} = v = \frac{\partial \varphi}{\partial y}$

2) Lin. Bernoulli: $\frac{\partial \varphi}{\partial t} + g\xi = 0$

see ex. 3.1

$$\frac{\partial \varphi}{\partial y} \sim \sin(kx-\omega t) \Rightarrow \varphi(x,y,t) = f(y)\sin(kx-\omega t)$$

$$\Rightarrow f'' - k^2 f = 0$$

$$\Rightarrow \varphi(x,y,t) = (Be^{ky} + Ce^{-ky})\sin(\ldots)$$

At $y \to -\infty$, $\varphi = 0 \Rightarrow C = 0$

$$\varphi = Be^{ky}\sin(\ldots)$$

$$\left.\frac{\partial \varphi}{\partial y}\right|_{y=0} = kB\sin(\ldots) = A\omega\sin(\ldots)$$

$$\Rightarrow B = \frac{\omega}{k}A$$

$$\left.\frac{\partial \varphi}{\partial t}\right|_{y=0} = -\omega B\cos(\ldots) = -\frac{\omega^2}{k}\cancel{A}\,\cancel{\cos}(\ldots)$$

$$+g\cancel{A}\,\cancel{\cos}(\ldots) = 0$$

=> disp. relation is $\omega^2 = gk$.

Speed at which the wave packet gen. by the storm moves:

$$c_g = \frac{d\omega}{dk}, \quad 2\omega\frac{d\omega}{dk} = g$$

$$\Rightarrow c_g = \frac{g}{2\omega} = \frac{g}{2\sqrt{gk}} = \frac{1}{2}\sqrt{\frac{g}{k}}$$

It will propag. at dist. x with veloc. c_g in t s:

$$x = c_g t \Rightarrow c_g = \frac{x}{t} = \frac{1}{2}\sqrt{\frac{g}{k}}$$

$$\Rightarrow \frac{x^2}{t^2} = \frac{g}{4k}, \quad \underbrace{k = \frac{gt^2}{4x^2}}_{\text{local wavenumber at } x \text{ and } t}$$

Subs. into disp. relation:

$$\omega = \frac{gt}{2x}$$

We know loc. freq. of this wavepacket when they arrive at the Brit. coast after some time t_1

$$\omega_1 = \frac{2\pi}{T_1} = \frac{gt_1}{2\underbrace{x}_{\text{our dist. from the storm}}} \to t_1 = \frac{4\pi x}{gT_1}$$

And a day later $t_2 = t_1 + 24h$:

$$\omega_2 = \frac{2\pi}{T_2} = \frac{gt_2}{2x} \to t_2 = \frac{4\pi x}{gT_2}$$

$$t_1 + 24h = \frac{4\pi x}{gT_1} + 86400s = \frac{4\pi x}{gT_2}$$

$$\to \frac{4\pi x}{g}\left(\frac{1}{T_2} - \frac{1}{T_1}\right) = 86400s$$

$$x = 86400\frac{T_1T_2}{T_1 - T_2}\frac{g}{4\pi} \approx 5059\,\text{km}$$